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Short Note

Effects of difference scheme type in high-order weighted compact nonlinear schemes

Taku Nonomura [a,*], Kozo Fujii [b]

[a] University of Tokyo, Yoshinodai 3-1-1 1815, Sagamihara, Kanagawa, Japan
[b] JAXA/Institute of Space and Astronautical Science, Yoshinodai 3-1-1 1813, Sagamihara, Kanagawa, Japan

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1. Introduction

The weighted compact nonlinear scheme (WCNS), developed by Deng and Zhang [1], is a high resolution scheme for flow fields including discontinuities. Deng and Zhang showed that WCNS has slightly higher resolution than the finite difference weighted essentially non-oscillatory scheme (WENO) [2] and similar discontinuity capturing capability as WENO. WCNS has the following three advantages compared with WENO: (1) various flux splitting methods can be used, e.g. Roe's flux difference splitting method (FDS) [3]; (2) interpolation of flow variables can be used despite the finite difference formulation; and (3) freestream and vortex preservation properties are very good on a wavy grid [4].

Thus far, the higher-order WCNSs have been developed by Nonomura et al. [5] and Zhang et al. [6] up to the ninth-order, while Zhang et al. proposed interpolating flux instead of the conservative variables used both in the original version and in the higher-order version developed by Nonomura et al. These researches showed that the resolution increases with an increasing order of accuracy and higher-order WCNSs are much more effective.

The WCNS procedure consists of three components [1]: (1) cell-node to cell-center weighted averaging interpolation of conservative variables, (2) flux evaluation at the cell-center and (3) cell-center to cell-node differencing. In the third component, various types of the cell-center to cell-node difference schemes are available. Deng and Zhang showed that the type of cell-center to cell-node difference scheme, explicit or tri-diagonal, does not affect the resolution of a fifth-order or fourth-order WCNS, because the weighted averaging interpolation (component 1) is dominant for its resolution [7].

* Corresponding author. Tel./fax: +81 42 759 8259.
E-mail address: nonomura@flab.isas.jaxa.jp (T. Nonomura).

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In the third component of the higher-order WCNS procedure, various types of the cell-center to cell-node difference schemes, e.g. explicit, tri-diagonal and penta-diagonal, are also available. However, the effects of the type of cell-center to cell-node difference scheme in a seventh- and ninth-order WCNSs have not been investigated. Unless a type of the cell-center to cell-node difference scheme changes the resolution of the scheme, an explicit scheme seems to be preferable because it is computationally cheap, easy to implement, and suitable for vectorization and parallelization, as suggested by Deng et al. [7] for the fifth-order WCNS.

This note presents a general form of the cell-center to cell-node difference scheme in a higher-order WCNS. The effects of the type of the cell-center to cell-node difference scheme, which is part of the seventh- and ninth-order WCNS, are investigated using Fourier analysis and one-dimensional problems, whereas the WCNS is formulated with interpolation of the conservative variables, as in Refs. [1,5]. In Section 2, the general form of the higher-order difference scheme in WCNS is explained and its coefficients are shown. In Section 3, wave resolutions of higher-order WCNSs with three types of the cell-center to cell-node difference scheme are analytically investigated with Fourier analysis. Then, in Section 4, higher-order WCNSs with three types of the cell-center to cell-node difference scheme are applied to one-dimensional problems, and the actual effects of type of the cell-center to cell-node difference scheme in a higher-order WCNS are verified. Section 5 concludes this note.

2. Generalized form of difference scheme in WCNS

The WCNS procedure in the original version and in the higher-order version developed by Nonomura et al. consists of three components as discussed before: (1) cell-node to cell-center weighted averaging interpolation of conservative variables, (2) flux evaluation at the cell-center and (3) cell-center to cell-node differencing. In this section, only the third component is presented. See Appendix A or Refs. [1,5,6] for the first and second components.

A following convection problem is considered here

$$\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial x} = 0, \tag{1}$$

where Q is a conservative variable and F is a flux. At every cell-node, this equation is semi-discretized in the uniform grid as follows

$$\left(\frac{\partial Q_j}{\partial t}\right) = -F'_j, \tag{2}$$

where F' denotes the approximation of the spatial derivative, and subscript j denotes the quantity on the jth grid point. For evaluating the F'_j, the third component of the WCNS procedure is conducted. We can give a general form of the compact and explicit cell-center to cell-node difference scheme as follows

$$\begin{aligned}\beta\widetilde{F}'_{j-2} + \alpha\widetilde{F}'_{j-1} + \widetilde{F}'_j + \alpha\widetilde{F}'_{j-1} + \beta\widetilde{F}'_{j-2} = {} & \frac{a}{\Delta x}(\widetilde{F}_{j+1/2} - \widetilde{F}_{j-1/2}) + \frac{b}{\Delta x}(\widetilde{F}_{j+3/2} - \widetilde{F}_{j-3/2}) + \frac{c}{\Delta x}(\widetilde{F}_{j+5/2} - \widetilde{F}_{j-5/2}) \\ & + \frac{d}{\Delta x}(\widetilde{F}_{j+7/2} - \widetilde{F}_{j-7/2}) + \frac{e}{\Delta x}(\widetilde{F}_{j+9/2} - \widetilde{F}_{j-9/2}),\end{aligned} \tag{3}$$

where $\widetilde{F}_{j+1/2}$ is the WCNS numerical flux on the cell-center which is computed through the first and second components; a, b, c, d, e, α and β are coefficients; and Δx is the uniform grid spacing.

The forth-order schemes are formulated as explicit $(\alpha = \beta = c = d = e = 0)$ and tri-diagonal schemes $(\beta = b = c = d = e = 0)$. The sixth-order schemes are formulated as explicit $(\alpha = \beta = d = e = 0)$ and tri-diagonal

Table 1
Coefficients of the generalized cell-center to cell-node difference scheme.

Coefficients	α	β	a	b	c	d	e	Required cell-center points
Fourth-order explicit	0	0	$\frac{9}{8}$	$-\frac{1}{24}$	0	0	0	4
Fourth-order tri-diagonal	$\frac{1}{22}$	0	$\frac{12}{11}$	0	0	0	0	2/global
Sixth-order explicit	0	0	$\frac{75}{64}$	$-\frac{25}{384}$	$\frac{3}{640}$	0	0	6
Sixth-order tri-diagonal	$\frac{9}{62}$	0	$\frac{63}{62}$	$\frac{17}{186}$	0	0	0	4/global
Eighth-order explicit	0	0	$\frac{1225}{1024}$	$-\frac{245}{3072}$	$\frac{49}{5120}$	$-\frac{5}{7168}$	0	8
Eighth-order tri-diagonal	$\frac{25}{118}$	0	$\frac{2675}{2832}$	$\frac{925}{5664}$	$-\frac{61}{28{,}320}$	0	0	6/global
Eighth-order penta-diagonal	$\frac{6114}{25{,}669}$	$\frac{183}{51{,}338}$	$\frac{23{,}400}{25{,}669}$	$\frac{14{,}680}{77{,}007}$	0	0	0	4/global
Tenth-order explicit	0	0	$\frac{19{,}845}{16{,}384}$	$-\frac{735}{8192}$	$\frac{567}{40{,}960}$	$-\frac{405}{229{,}376}$	$\frac{35}{294{,}912}$	10
Tenth-order tri-diagonal	$\frac{49}{190}$	0	$\frac{12{,}985}{14{,}592}$	$\frac{78{,}841}{364{,}800}$	$-\frac{343}{72{,}960}$	$\frac{129}{851{,}200}$	0	8/global
Tenth-order penta-diagonal	$\frac{96{,}850}{288{,}529}$	$\frac{9675}{577{,}058}$	$\frac{683{,}425}{865{,}587}$	$\frac{505{,}175}{1{,}731{,}174}$	$\frac{69{,}049}{8{,}655{,}870}$	0	0	6/global

schemes($\beta = c = d = e = 0$). The eighth-order schemes are formulated as explicit ($\alpha = \beta = e = 0$), tri-diagonal ($\beta = d = e = 0$) and penta-diagonal schemes ($c = d = e = 0$). The tenth-order schemes are also formulated as explicit ($\alpha = \beta = 0$), tri-diagonal ($\beta = e = 0$) and penta-diagonal schemes ($d = e = 0$). The value of coefficients for these schemes is summarized in Table 1. The tri-diagonal and penta-diagonal difference schemes are formulated as compact (implicit). These compact schemes need matrix inversions.

Table 1 includes the required number of cell-center points for differencing. These schemes need more points with increasing order of accuracy, while a wider stencil becomes load for parallelization. Although compact scheme requires a less number of cell-center points, they are less suitable for parallelization because they are global schemes, which need the data of all points in matrix inversions. Table 1 also shows whether the global scheme or not in the column of the required number of cell-center points.

In this study, the fifth-order WCNS is computed with fifth-order weighted interpolation and sixth-order difference schemes. WCNS5E and WCNS5T denote fifth-order WCNSs with sixth-order explicit and tri-diagonal difference schemes, respectively. The seventh-order WCNS is computed with seventh-order weighted interpolation and eighth-order difference schemes. WCNS7E, WCNS7T and WCNS7P denote the seventh-order WCNSs with eighth-order explicit, tri-diagonal and penta-diagonal difference schemes, respectively. The ninth-order WCNS is computed with ninth-order weighted interpolation and tenth-order difference schemes. WCNS9E, WCNS9T and WCNS9P denote the ninth-order WCNSs with tenth-order explicit, tri-diagonal and penta-diagonal difference schemes, respectively.

3. Effects of the type of difference scheme for ideal weights using Fourier analysis

In this section, the wave resolutions of WCNS for ideal weights are investigated using Fourier analysis as in the Refs. [8,1,6]. The schemes investigated are WCNS5E, WCNS5T, WCNS7E, WCNS7T, WCNS7P, WCNS9E, WCNS9T, WCNS9P, and the sixth-order Pade type compact difference scheme [8] as a reference.

In this analysis, a modified wave number ω' for WCNS is computed with Fourier analysis as in the Ref. [6]. The modified wave number ω' denotes the numerically evaluated value of the wave number ω in the numerical differential of $f = \exp(i\omega x_j)$ as follows:

$$\left(\frac{\partial f}{\partial x}\right)_{num} = i\omega' \exp(i\omega x_j), \tag{4}$$

where the subscript *num* denotes the numerically evaluated quantity; i denotes the $\sqrt{-1}$; j denotes the quantity on the jth grid point; and $x_j = j\Delta x$. In this analysis, ideal weights are used instead of the nonlinear weights, because it is difficult to evaluate the nonlinear scheme with Fourier analysis.

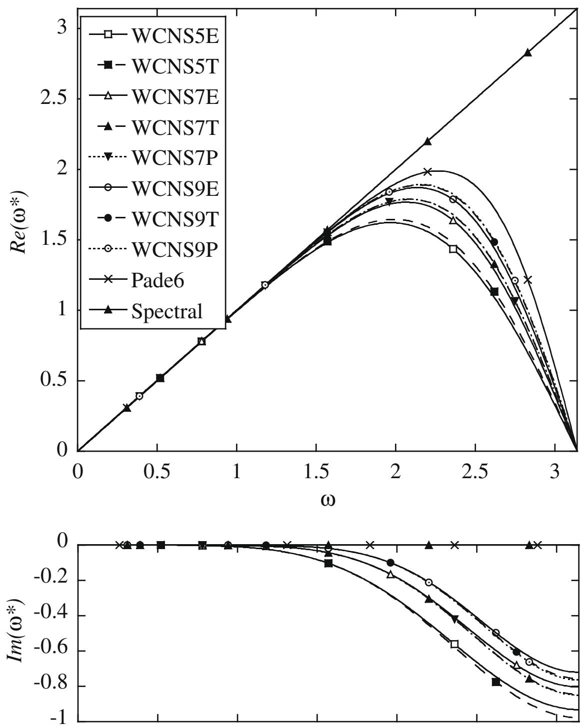

Fig. 1. Fourier analysis of the higher-order WCNSs.

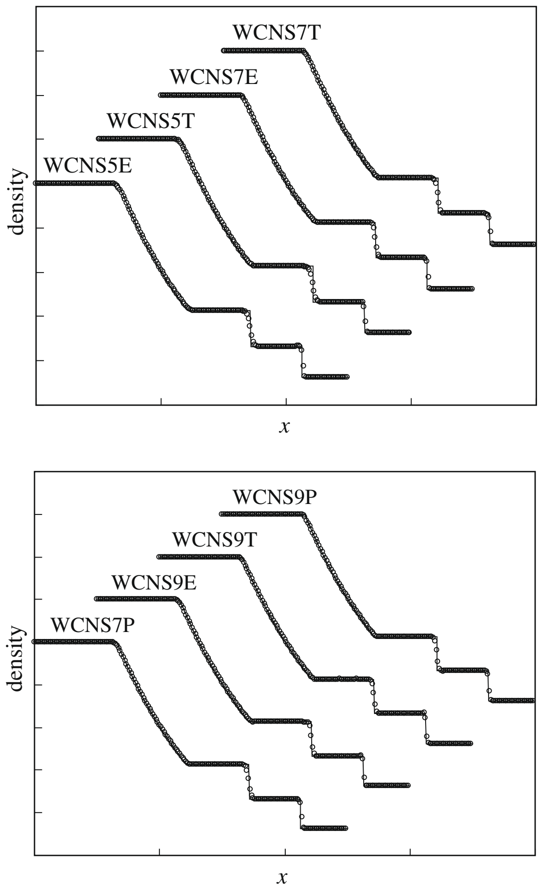

Fig. 2. Density of Sod's problem with higher-order WCNSs. (solid line: exact solution [11]).

The results of this analysis are shown in Fig. 1. The imaginary parts of ω^* for WCNSs are not zero unlike in the compact scheme, because the WCNS is an upwind scheme, while the compact scheme is a central scheme.

Fig. 1 shows that differences among the explicit, tri-diagonal and penta-diagonal schemes changes the modified wave number slightly, while the order of accuracy changes the resolutions significantly. Therefore the result shows that the type of cell-center to cell-node difference scheme in a higher-order WCNS does not change the resolution significantly. The Fourier analysis, using ideal weights, shows that the explicit scheme, which can be efficiently implemented, is better for a higher-order WCNS as suggested by Deng et al. [7] for the fifth-order WCNS.

4. Effects of the type of difference scheme using one-dimensional problems

One-dimensional numerical tests of the higher-order WCNSs were conducted. In these tests, one-dimensional Euler equations were computed. These tests adopted Roe's FDS for the flux evaluation and the third-order total variation diminishing Runge–Kutta scheme [9] for time integration. First, Sod's problem [10] was computed with a higher-order WCNS for inves-

Table 2
Discontinuity thickness of Sod's problem with the higher-order WCNSs.

Scheme	Shock	Contact
WCNS5E	2.026	4.068
WCNS5T	2.027	4.068
WCNS7E	2.035	3.281
WCNS7T	2.163	3.376
WCNS7P	2.157	3.377
WCNS9E	1.639	3.035
WCNS9T	1.565	2.922
WCNS9P	1.800	3.036

tigating the effects of the type of difference scheme on the discontinuity capturing capability and the resulting thickness of discontinuities. The problem is formulated as the initial value problem as presented in Refs. [10,1].

In this test case, grid points were set to 201, the CFL number was set to 0.6, and the time was integrated to time $t = 2.0$. Schemes, WCNS5E, WCNS5T, WCNS7E, WCNS7T, WCNS7P, WCNS9E, WCNS9T and WCNS9P were examined. Fig. 2 shows the computational results. The solid line indicates the exact solution [11]. All schemes resolved shock waves without any remarkable numerical oscillations, although tiny over-shoots or under-shoots were observed. These results show that the type of the cell-center to cell-node difference scheme does not affect the shock capturing capability.

Here, discontinuity thickness in this problem is discussed. The discontinuity thickness n_{dis} is computed as follows,

$$n_{dis} = \frac{\Delta\rho}{\max\left(\rho_j - \rho_{j+1}\right)}, \tag{5}$$

where $\Delta\rho$ is the density jump across the discontinuity. The computed results are shown in Table 2. As the order of accuracy increases, the thickness of the contact surface decreases. The thickness of the shock wave for ninth-order WCNSs are smaller than fifth and seventh-order WCNSs, while those of fifth-order and seventh-order are almost the same. However, the type of cell-center to cell-node difference scheme does not change discontinuity thickness except for the thickness of the shock wave in ninth-order WCNSs. Therefore, these results show that the type of the cell-center to cell-node difference scheme basically does not affect the discontinuity thickness. The difference in the shock wave thickness among WCNS9E, WCNS9T and WCNS9P is caused by a difference of slight over-shoots (4% magnitude of jump) near the shock wave. Thus, these results do not show the clear advantages of the WCNS9T compared with WCNS9E and WCNS9P.

The shock-entropy wave interaction (Shu–Osher) problem [12] was computed to investigate the effect of the type of the difference scheme on the resolution of the short (high-frequency) waves for the flow fields including discontinuity. This problem is also formulated as the initial value problem, as presented in Refs. [12,6].

There were 201 grid points used in the test. The CFL number was set to 0.6 and the time was integrated to $t = 1.8$. The schemes used in this test were same as those used for Sod's problem. Fig. 3 shows the computational results. The solid line shows the reference solution, which was computed with 1601 grid points by WCNS9E. Comparison of WCNS5E, WCNS7E and

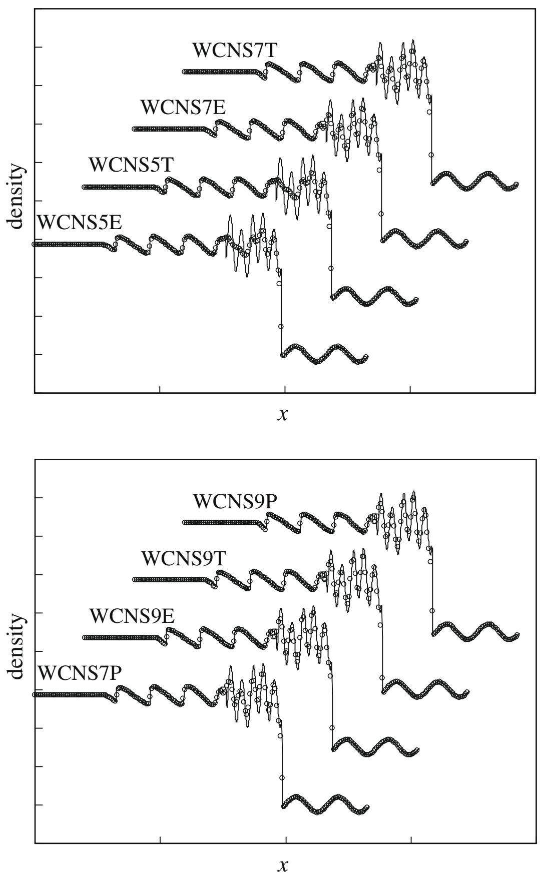

Fig. 3. Density of shock-entropy wave interaction problem with the higher-order WCNSs. (solid line: reference solution).

WCNS9E shows that the higher-order WCNS resolves the short waves of density more precisely. Thus the higher resolution can be obtained with the higher-order WCNS. However, comparison of WCNS7E, WCNS7T and WCNS7P, or WCNS9E, WCNS9T and WCNS9P, shows that resolutions of the same order schemes are almost similar. Therefore, Fig. 3 shows that the type of the cell-center to cell-node difference scheme does not affect the resolution of WCNS for short waves. This trend corresponds to the Fourier analysis in the Section 3.

In this section, an uniform grid is used. Since WCNS is a difference scheme, it works well in a non-uniform grid using coordinate transformation as far as the non-uniform grid is smooth enough. Additionally, it is shown that WCNS has very good freestream and vortex preservation properties compared with WENO on a three dimensional wavy grid [4]. On the other hand, it is difficult to apply WCNS to a non-smooth grid with keeping its accuracy.

5. Conclusion

This note introduce a general form of the cell-center to cell-node difference scheme in the higher-order WCNS. Then, Fourier analysis of the higher-order WCNS is conducted. It shows that the analytical resolution of WCNS is not affected by the type of cell-center to cell-node difference scheme. Then two one-dimensional problems, Sod's problem and the shock-entropy wave interaction problem are numerically examined. For the former problem, the type of cell-center to cell-node difference scheme does not change the shock capturing capability and the discontinuity thickness of the WCNS. For the latter problem, the type of cell-center to cell-node difference scheme does not change the resolution of short waves. These results imply that the explicit form of cell-center to cell-node difference scheme is preferable because the explicit form is the computationally cheapest, easy to implement, and suitable for vectorization and parallelization, as suggested for the fifth-order WCNS by Deng et al. [7]. These results imply that the weighted averaging interpolation in the WCNS procedure is still dominant for the resolutions of a higher-order WCNS.

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Appendix A. First and second components in WCNS procedure

In this appendix, the first and second components in WCNS procedure are explained in brief.

In the fisrt component, the variables Q_j on the grid point are interpolated to that $Q^L_{j+1/2}, Q^R_{j+1/2}$ on left side and right side of the cell-center with upwinding stencils. Hereafter, the construction of $(2r-1)$th-order $Q^L_{j+1/2}$ are only noted, while we can compute $Q^R_{j+1/2}$ symmetrically. Following an one-point upwind biased $(2r-1)$ points stencil $[j-r+1,\dots,j,\dots,j+r-1]$ is used in $(2r-1)$th-order interpolation. In this stencil, conservative variables Q are transformed into characteristic variables $\mathcal{Q}$:

$$\mathcal{Q}_{l,m} = l_{j,m} Q_l \quad (l = j-r+1,\dots,j,\dots,j+r-1), \tag{6}$$

where $\mathcal{Q}_{l,m}$ denotes the mth characteristic variable, and $l_{j,m}$ denotes the mth left eigenvector of the matrix $A = \partial F/\partial Q$ on the jth grid point.

Then r-points sub-stencils are constructed. The kth $(k = 1,2,\dots,r)$ sub-stencil consists of $[j+k-r, j+k-r+1,\dots,j+k-1]$. The rth-order interpolation to cell-center using the kth sub-stencil is computed as a linear combination of the characteristic variables as follows:

$$\mathcal{Q}^L_{j+1/2,k,m}\left(\mathcal{Q}_{j-r+k,m}, \mathcal{Q}_{j-r+k+1,m}, \dots, \mathcal{Q}_{j+k-1,m}\right). \tag{7}$$

Then weighted value is computed as follows:

$$\mathcal{Q}^L_{j+1/2,m} = \sum_{k=1}^{r} w_{k,m} \mathcal{Q}^L_{j+1/2,k,m}, \tag{8}$$

where $w_{k,m} (k = 1,\dots,r)$ are nonlinear weights for mth characteristic variables. For smooth region, $w_{k,m}$ turn to be ideal weights for the $(2r-1)$th order interpolation, while, for non-smooth region, $w_{k,m}$ are determined to suppress the oscillations. For more detail, see Refs. [1,5,6].

Finally, the characteristic form of the interpolated value is transformed into the conservative form:

$$Q^L_{j+1/2} = \sum_m \mathcal{Q}^L_{j+1/2,m} r_{j,m}, \tag{9}$$

where $r_{j,m}$ denotes the mth right eigenvector of the matrix $A = \partial F/\partial Q$ on the jth grid point. In this manner, $Q^L_{j+1/2}$ is obtained.

In the second component fo the WCNS procedure, $F'_{j+1/2}$ is computed from $Q^L_{j+1/2}$ and $Q^R_{j+1/2}$ as follows:

$$\widetilde{F}_{j+1/2} = F_{FS}(Q^L_{j+1/2}, Q^R_{j+1/2}), \tag{10}$$

where F_{FS} is a function of arbitrary flux splitting schemes, such as the Roe's FDS, van Leer's flux vector splitting (FVS) [13] or the advection upstream splitting method (AUSM) [14].

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